

MINIMAL FIBRATIONS OF HYPERBOLIC 3-MANIFOLDS

JOEL HASS

ABSTRACT. There are hyperbolic 3-manifolds that fiber over the circle but that do not admit fibrations by minimal surfaces. These manifolds do not admit fibrations by surfaces that are even approximately minimal.

1. INTRODUCTION

Thurston showed that a 3-manifold that is a bundle over S^1 with fibers a closed surface of genus $g \geq 2$, and with pseudo-anosov monodromy, admits a hyperbolic metric, and conjectured that all hyperbolic 3-manifolds were finitely covered by such bundles [21]. This conjecture was recently proved in work of Agol [2], and indicates that understanding the geometry of such manifolds is central to understanding the geometry of general hyperbolic 3-manifolds.

In this article we give the first examples of hyperbolic 3-manifolds that fiber over the circle but do not admit fibrations in which each fiber is a minimal surface. We further show that these manifolds do not admit fibrations that are even approximately minimal. No obstructions to the existence of minimal fibrations of such manifolds were previously known.

The obstruction to a minimal fibration comes from the geometry of a hyperbolic manifold near a short geodesic. Thurston's characterization of hyperbolic surface bundles points the way to the construction of hyperbolic surface bundles that contain arbitrarily short, null-homologous geodesics. Near a short geodesic the geometry of a hyperbolic 3-manifold resembles that of a cusp. Direct estimates show that the area of an incompressible surface going far into a cusp is larger than that of a homotopic surface that penetrates the cusp less deeply. One consequence is that such a surface cannot be least area in its homology class. On the other hand, a leaf of a minimal fibration has no greater area than any homologous surfaces [20]. We construct examples where fibers must go arbitrarily deeply into a neighborhood of a short geodesic, but can be homotoped out of the neighborhood. A minimal fibration cannot exist in these manifolds. The relation between minimal surfaces and short geodesics has been studied by Breslin [5]. In this paper we have not attempted to obtain explicit estimates on the lengths of shortest geodesics that provide obstructions. Some explicit estimates have been given recently by Huang and Wang [10].

An embedded, orientable, incompressible surface in a closed, orientable, Riemannian 3-manifold is homotopic to a surface of least area [19, 17], and this surface is either embedded or double covers an embedded 1-sided surface [7]. It follows

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that in a fibered hyperbolic 3-manifold there is an embedded minimal surface homotopic to a fiber. It is not known when the fibers can ever be isotoped so that all are minimal, giving a minimal fibration.

Minimal foliations of non-compact hyperbolic manifolds do exist, so the obstruction to a minimal fibration is not local. Hyperbolic space itself can be minimally foliated in many different ways. One such foliation is given by totally geodesic planes, whose limit sets form parallel meridian circles on the sphere at infinity, foliating the 2-sphere with its two poles removed by meridians. A large class of minimal foliations can be constructed by perturbing the meridians to curves that remain transverse to longitudes, and then taking a family of least area planes that limit to these curves. Such least area planes exist [3], and this family forms a foliation, since the least area planes spanning disjoint curves are disjoint and an application of the maximum principle shows that there are no gaps between planes. This construction can be done equivariantly under a hyperbolic translation that preserves the two poles, giving a minimal fibration over the circle with planar fibers. No known construction gives minimal fibers of finite area.

Corollary 1.1 shows that many hyperbolic 3-manifolds that fiber over S^1 do not admit minimal fibrations. This follows from Theorem 1.1, which shows the non-existence of fibrations whose fibers are even approximately minimal. We now make this concept precise.

Let M be a smooth Riemannian manifold and $X \subset M$ a compact surface in M , either closed or with boundary. We set

$$\mathcal{I}_X = \inf\{\text{Area}(G) \mid G \subset M \text{ is a smooth surface homologous to } X \text{ (rel } \partial G)\}.$$

A surface F is called *area minimizing* if for any compact subsurface X of F ,

$$\text{Area}(X) = \mathcal{I}_X.$$

A surface $F \subset M$ is (h, λ) -*quasi-area-minimizing* if

- (1) The mean curvature H of F satisfies $|H| < h$,
- (2) For any compact subsurface $X \subset F$, $\text{Area}(X) < \lambda \cdot \mathcal{I}_X$.

Theorem 1.1. *Let $a < 1$ and $b < 2$ be constants. There are hyperbolic 3-manifolds that are surface bundles over S^1 and that admit no fibration whose fibers are (a, b) -quasi-area-minimizing.*

We prove Theorem 1.1 in Section 4.

Corollary 1.1. *There are hyperbolic 3-manifolds that fiber over S^1 that do not admit a minimal fibration.*

Proof. The surfaces in a minimal fibration are each area minimizing in their homology class [20]. This implies that they are (a, b) -quasi-area-minimizing surfaces for any $a > 0$ and any $b > 1$. In particular each fiber is a $(1, 2)$ -quasi-area-minimizing surface, and it follows that the manifolds in Theorem 1.1 admit no minimal fibrations. \square

The existence of minimal fibrations is closely related to the question of whether there exist non-isolated minimal surfaces in hyperbolic 3-manifolds, and to the existence of unstable minimal surfaces.

A foliation of a Riemannian 3-manifold with 2-dimensional leaves is *taut* if each leaf intersects a closed transversal curve. Fibrations give one class of examples of

taut foliations. Taut foliations were studied by Novikov, who showed among other results that each leaf of such a foliation is incompressible [14]. Sullivan showed that a smooth foliation of a 3-manifold is taut if and only if there is a Riemannian metric on the manifold in which each leaf is a minimal surface [20]. A consequence of Sullivan's theorem is that a surface bundle over S^1 admits some Riemannian metric in which each fiber is minimal. Corollary 1.1 shows that Sullivan's construction is often not compatible with a hyperbolic metric.

The results in this paper originate in conversations with Bill Thurston at the 1984 Durham Symposium on *Kleinian groups, 3-Manifolds, and Hyperbolic Geometry*. Other results concerning minimal surfaces in cusps can be found in [6, 12, 13, 16].

2. SURFACES IN CUSPS

We review some standard facts about the geometry of cusps of hyperbolic 3-manifolds. An *ideal hyperbolic cusp* C is a hyperbolic 3-manifold homeomorphic to the product $T^2 \times \mathbb{R}$, obtained as the quotient of H^3 by a parabolic subgroup Γ of $\mathrm{PSL}(2, C)$ isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. In the upper half-space model of H^3 the generators of Γ act as translations of the xy -plane. A fundamental domain for the cusp is $Q \times (0, \infty)$, where Q is a parallelogram on the xy -plane. The cusp is foliated by flat horotori T_s , with T_s covered by the horosphere $\{z = s\}$.

Conjugating Γ by the isometry $z \rightarrow \lambda z$, $\lambda \in \mathbb{R}^+$, takes the horosphere $\{z = s\}$ to the horosphere $\{z = \lambda s\}$. We can use this conjugacy to arrange that the horotorus whose shortest non-trivial curve has length one lifts to the plane $\{z = 1\}$, so that T_1 has injectivity radius $1/2$. The lengths of curves on the horotori T_s decrease linearly with s , so that the shortest curve on T_s has length $1/s$, or equivalently, the injectivity radius of T_s satisfies $i_s = 1/(2s)$ for all s . Let $C_{[a,b]}$ denote the portion of the cusp consisting of T_s with $a \leq s \leq b$ and $C_{[a,\infty)}$ the end of the cusp cut off by T_a .

Lemma 2.1. *Let D be a smooth disk in an ideal cusp C with $\partial D \subset T_s$ whose boundary has length l . Then either $D \subset C_{[s, s+(sl^2)/4]}$ or D has an interior point where its mean curvature satisfies $|H| \geq 1$.*

Proof. We work in the upper half-space model. The disk D lifts to a disk \tilde{D} in H^3 and its boundary curve lifts to a curve γ in the horosphere $\{z = s\}$. Let E be a disk in the horotorus $\{z = s\}$ of radius $l/2$ (in the induced flat horotorus metric), centered on a point of γ . Since γ lies on the horotorus and has hyperbolic length l , its length in the flat horotorus metric is also l , and it lies in the interior of E . In the Euclidean metric on the upper-half-space, E has radius $ls/2$ and is the intersection of a horoball B of H^3 with the horosphere $\{z = s\}$. A calculation shows that the horoball B has Euclidean height $h = s + (sl^2)/4$, as indicated in Figure 1.

If the interior of \tilde{D} meets $\{z = s\}$, then it tangentially meets some horosphere $\{z = s'\}$ with $s' < s$ and s' minimal. Thus it meets $\{z = s'\}$ tangentially, without crossing it. Since the mean curvature of a horosphere is 1, the mean curvature of \tilde{D} at the point of tangency satisfies $|H| \geq 1$. It follows that if the mean curvature of \tilde{D} is less than 1 then $D \subset C_{[s,\infty)}$.

If \tilde{D} is not contained in B then it is contained in a largest horoball B' such that $B \subset B'$ and \tilde{D} meets $\partial B'$ without crossing it. Again \tilde{D} has mean curvature $|H| \geq 1$ at the point where it meets $\partial B'$. Thus if the mean curvature of \tilde{D} is less

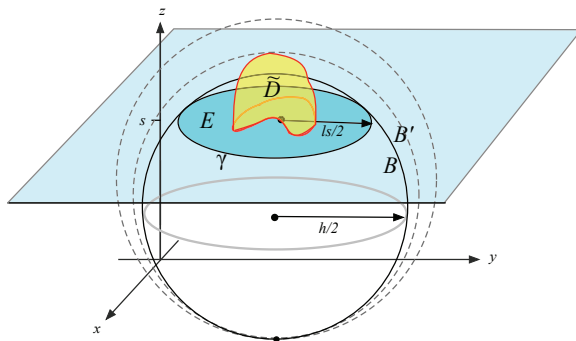


FIGURE 1. A minimal disk bounded by a curve of hyperbolic length l that lies on the horosphere $\{z = s\}$ is contained inside a horoball of height $h = s + (sl^2)/4$. Indicated distances are Euclidean.

than one, then it lies in the slab $\{s \leq z \leq s + (sl^2)/4\}$ and D lies in its quotient $C_{[s, s+(sl^2)/4]}$. \square

Lemma 2.2. *Let D be a smooth disk in an ideal cusp C with $\partial D \subset T_s$ and $\text{length}(\partial D) = l$ with $l < 2i_s$ and $s \geq 1/2$. Then either $D \subset C_{[s, 2s]}$ or D has an interior point where its mean curvature satisfies $|H| \geq 1$.*

Proof. Since $i_s = 1/(2s)$ and $l < 2i_s$,

$$s + \frac{sl^2}{4} < s + s(i_s)^2 = s + \frac{1}{4s} \leq s + 1/2 \leq 2s.$$

\square

We now consider a complete, finite-volume hyperbolic 3-manifold M with a cusp C . The fundamental group of the cusp is isomorphic to a $\mathbb{Z} \oplus \mathbb{Z}$ -subgroup of $\pi_1(M)$ generated by parabolic elements. With the upper-half space model representing the universal cover of M , this $\mathbb{Z} \oplus \mathbb{Z}$ -subgroup can be conjugated to fix infinity, and thus act as translations. The covering of M corresponding to this subgroup is an ideal cusp C , which is foliated by horotori whose injectivity radius approaches 0 as they approach the end of the cusp. As before we parametrize the horotori of $C_{(s, \infty)}$ so that T_s has injectivity radius $1/(2s)$. The projection of C to M is injective on $C_{(m, \infty)}$ for some $m \leq 1$ [1]. Thus the cusp $C_{(m, \infty)}$ of M is isometric to a submanifold of an ideal cusp cut off by a horotorus. The *maximal horotorus* T_m lies in the boundary of the cusp. It is self-tangent at some number of points, but is the limit of embedded horotori in the cusp. For $s > m$ the cusp $C_{(m, \infty)} \subset M$ is isometric to an end of an ideal cusp and so the results and terminology of Lemmas 2.1 and 2.2 apply in M .

The next lemma implies that a (1,2)-quasi-quasi-area-minimizing disk with boundary on T_s does not reach too far into a cusp, as long as $s \geq 4\text{Area}(T_1)$. Note that the quantity $\text{Area}(T_1)$ is always at least $\sqrt{3}/2$, which is obtained when T_1 is the quotient of \mathbb{R}^2 by two translations of length one at an angle of $\pi/3$.

Lemma 2.3. *Let M be a finite volume hyperbolic manifold with cusp C , s a constant such that $s \geq 4\text{Area}(T_1)$, and D a (1,2)-quasi-area-minimizing disk in M with $\partial D \subset T_s$. Then $D \subset C_{[s, 4s]}$.*

Proof. By lifting D to the ideal cusp covering space of M , we see as in Lemma 2.2 that if $|H| < 1$ then D lies within the cusp end $C_{[s,\infty)}$.

Assume for contradiction that D is not contained in $C_{[s,4s)}$. If there exists a $z \in [s, 2s)$ with $\text{length}(D \cap T_z) < 2i_z$ then $2z < 4s$ and Lemma 2.2 implies that $D \subset C_{[s,2z]} \subset C_{[s,4s)}$. Thus $\text{length}(D \cap T_z) \geq 2i_z = 1/z$ for all $z \in [s, 2s)$. The area of D in $C_{[s,2s)}$ can then be bounded below by the co-area formula. In the upper-half space model, this gives

$$\text{Area}(D \cap C_{[s,2s)}) \geq \int_s^{2s} \text{length}(D \cap T_z) \frac{1}{z} dz \geq \int_s^{2s} \frac{1}{z^2} dz = \frac{1}{2s}.$$

So a disk not contained in $C_{[s,4s)}$ has area larger than $1/(2s)$.

Now ∂D is a null-homotopic simple curve in T_s , and therefore bounds a disk on T_s whose area is strictly less than $\text{Area}(T_s) = \text{Area}(T_1)/s^2$, implying that

$$\mathcal{J}_D < \text{Area}(T_1)/s^2.$$

Since we assumed $4\text{Area}(T_1) \leq s$, we have

$$2\mathcal{J}_D < \frac{2\text{Area}(T_1)}{s^2} \leq \frac{1}{2s} \leq \text{Area}(D \cap C_{[s,2s)}).$$

Since D is a (1,2)-quasi-area-minimizing disk, this gives a contradiction, showing that $D \subset C_{[s,4s)}$. \square

We now analyze the intersection of a (1,2)-quasi-area-minimizing incompressible surface with a cusp.

Lemma 2.4. *Let M be a finite volume hyperbolic manifold with cusp C , and $T_s = \partial C_{[s,\infty)}$ with $s \geq s_0 = 4\text{Area}(T_1)$. Suppose that $F \subset C_{[s,\infty)}$ is a smooth, compact, embedded, incompressible, (1,2)-quasi-area-minimizing surface with boundary. Then $F \cap T_{4s} = \emptyset$.*

Proof. Note that F is separating since it is homotopic into T_s . Moreover F must have boundary, as otherwise it would meet a lowest horotorus on the cusp at a point where $H \geq 1$. So the curves of $F \cap T_s$ separate T_s into two subsurfaces. Each of these has area at most

$$\text{Area}(T_s) = \frac{\text{Area}(T_1)}{s^2}.$$

The surface F is homologous to one of these subsurfaces (rel boundary). Comparing areas and using the assumption that $4\text{Area}(T_1) \leq s$, we see that

$$\mathcal{J}_F < \text{Area}(T_s) \leq \frac{\text{Area}(T_1)}{s^2} \leq \frac{1}{4s}.$$

Suppose that for all $z \in [s, 2s]$, the length of the intersection $F \cap T_z$ satisfies $\text{length}(F \cap T_z) \geq 2i_z = 1/z$. Then the area of F in the cusp region $C_{[s,2s)}$ can be bounded below with the co-area formula,

$$\text{Area}(F \cap C_{[s,2s)}) \geq \int_s^{2s} \text{length}(F \cap T_z) \frac{1}{z} dz \geq \int_s^{2s} \frac{1}{z^2} dz = \frac{1}{2s}.$$

Since F is (1,2)-quasi-area-minimizing,

$$\frac{1}{2s} \leq \text{Area}(F) < 2\mathcal{J}_F < \frac{1}{2s},$$

which is a contradiction. So there exists $z \in [s, 2s)$ with $\text{length}(F \cap T_z) < 2i_z = 1/z$. For this z , $F \cap T_z$ consists of a collection of null-homotopic curves, each shorter than

$2i_z$. Since F is incompressible, each of these bounds a disk in M . Lemma 2.2 then implies that each such disk is contained in $C_{[z,2z]}$ and therefore that $F \subset C_{[s,2s]} \subset C_{[s,4s]}$. We conclude that F does not meet T_{4s} . \square

Lemma 2.5. *Let M be a hyperbolic 3-manifold with cusp C and $F \subset M$ a smooth, compact, embedded, incompressible, (1,2)-quasi-area-minimizing surface in M . If $s \geq 16\text{Area}(T_1)$ then $F \cap T_s = \emptyset$.*

Proof. Let $s_0 = 4\text{Area}(T_1)$. Then the surface $F \cap C_{[s_0, \infty)}$ is a smooth, compact, embedded, (1,2)-quasi-area-minimizing surface with boundary. It is incompressible because F is incompressible in M , and a disk in F whose boundary lies in $C_{[s_0, \infty)}$ has interior that also lies in $C_{[s_0, \infty)}$, as otherwise its mean curvature would be somewhere greater than one. Therefore $F \cap C_{[s_0, \infty)}$ satisfies the hypothesis of Lemma 2.4 and $F \cap T_{4s_0} = \emptyset$. \square

We will also use a bound that holds for minimal (mean curvature zero) surfaces. We show that these surfaces have area that grows linearly with the parameter s .

Lemma 2.6. *Let M be a hyperbolic 3-manifold with cusp C , $s \geq 8\text{Area}(T_1)$, and $F \subset C_{[s, 2s]}$ a smooth, properly embedded minimal surface with $\partial F \cap T_s \neq \emptyset$ and $\partial F \cap T_{2s} \neq \emptyset$. Then either $F \cap T_u = \emptyset$ for some $u \in (s, 2s)$ or $\text{Area}(F) > 3/(32s)$.*

Proof. By standard monotonicity estimates, as in [3], the area of a minimal surface passing through the centre of a ball of radius r in hyperbolic space is at least as large as a hyperbolic disk of radius r , namely $4\pi \sinh^2(r/2)$. We use this to make some rough approximations to the area of F , assuming that $F \cap T_u \neq \emptyset$ for $u \in (s, 2s)$.

A ball of Euclidean diameter one in the upper-half space model that lies between $z = s$ and $z = s + 1$ can be embedded in the cusp between T_s and T_{s+1} so its Euclidean center lies at any point in $T_{s+1/2}$. Such a ball has hyperbolic diameter given by

$$\int_s^{s+1} 1/z \, dz = \ln(s+1) - \ln(s) = \ln(1 + 1/s) \geq \frac{1}{s} - \frac{1}{2s^2}.$$

Since $s \geq 8\text{Area}(T_1) \geq 4\sqrt{3} > 4$ we have that the hyperbolic diameter is greater than $1/(2s)$. Thus the radius of this ball is at least $1/(4s)$ and the monotonicity estimate tells us that a minimal surface passing through the ball's center has area inside the ball of no less than

$$4\pi \sinh^2\left(\frac{1}{4s}\right) > \frac{\pi}{16s^2} > \frac{3}{16s^2}.$$

We can apply this estimate to s disjoint balls in the cusp of Euclidean radius one, with one ball lying between each adjacent pair of the planes $z = s, z = s+1, \dots, z = 2s$, and with each ball centered on a point of F . If F does not miss an intermediate T_u , then this results in a lower bound on the area of F of

$$\text{Area}(F) > \left(\frac{3}{16}\right)\left(\frac{1}{s^2} + \frac{1}{(s+1)^2} \cdots + \frac{1}{(2s)^2}\right) > \frac{3}{32s}$$

as claimed. \square

Lemma 2.7. *Let M be a hyperbolic 3-manifold with cusp C , $s \geq 14\text{Area}(T_1)$, and $F \subset C_{[s, 2s]}$ a smooth, properly embedded, separating, area-minimizing surface with $\partial F \cap T_s \neq \emptyset$. Then $F \cap T_u = \emptyset$ for some $u \in [s, 2s]$.*

Proof. If $F \cap T_u \neq \emptyset$ for $u \in (s, 2s]$, then Lemma 2.6 implies that

$$\frac{3}{32s} < \text{Area}(F).$$

Since F is area minimizing we can bound its area by the homologous surface formed by parts of each torus on $\partial C_{[s, 2s]}$. We can compare its area to that of a homologous surface formed by a subsurface of T_s , having area at most $\text{Area}(T_s)$, and if F also has boundary on T_{2s} , an additional subsurface of T_{2s} whose area is at most $\text{Area}(T_{2s})$.

$$\text{Area}(F) \leq \text{Area}(T_s) + \text{Area}(T_{2s}) = \frac{5\text{Area}(T_1)}{4s^2}.$$

Combining these gives

$$\frac{3}{32s} < \frac{5 \text{Area}(T_1)}{4s^2}$$

implying that $s < (40/3)\text{Area}(T_1)$. This contradicts our assumption on s , implying that $F \cap T_u = \emptyset$ for some $u \in [s, 2s]$. \square

3. BUNDLES WITH SHORT GEODESICS

In this section we use Thurston's Dehn Surgery Theorem [21, 4, 8, 13] to construct a sequence of hyperbolic 3-manifolds M_j that fiber over the circle and limit to a hyperbolic 3-manifold with a cusp. These manifolds contain embedded geodesics that are homotopic into a fiber and whose length approaches zero as $j \rightarrow \infty$.

The complement of an open neighborhood of a simple closed geodesic in a closed hyperbolic 3-manifold M_0 is the interior of a manifold M with torus boundary. The manifold M is atoroidal and irreducible and therefore satisfies the hypothesis of Thurston's geometrization theorem for Haken manifolds. The absence of essential spheres and tori can be seen by considering the lift of such a surface to the complement of a collection of geodesic lines in the universal cover of M . Alternately spheres and non-peripheral tori can be ruled out by noting that it is possible to explicitly construct a negatively curved metric on the complement [11]. It follows that M has a complete, finite volume hyperbolic metric with a single cusp.

The notion of geometric convergence allows for comparing a non-compact manifold to a sequence of compact manifolds that resemble it on increasingly large subsets. Given $x \in M$ let $B_{r_j}(M, x)$ be the set of points in M whose distance from x is at most r_j . Then a sequence of Riemannian manifolds (M_i, g_i) is said to *geometrically converge* to (M, g) if for any fixed x and sequence $r_j \rightarrow \infty$, there are points $x_j \in M_j$ and a sequence of maps $f_j : B_{r_j}(M, x) \rightarrow B_{r_j}(M_j, x_j)$ that converge smoothly to isometries, meaning that on each compact region in M , the pulled back Riemannian metrics $f_j^*(g_j)$ on M converge smoothly to the hyperbolic metric on M . See Figure 2.

We construct a geometrically convergent sequence of manifolds M_j that fiber over the circle.

Lemma 3.1. *There exists a sequence of hyperbolic 3-manifolds $M_j, j = 1, 2, \dots$ with the following properties:*

- (1) *For each $j \geq 1$, M_j fibers over the circle.*
- (2) *The manifolds M_j converge geometrically to a cusped hyperbolic manifold M as $j \rightarrow \infty$.*
- (3) *There is a sequence of closed, null-homologous geodesics $\gamma_j \subset M_j$, homotopic into a fiber of M_j , with $\lim_{j \rightarrow \infty} \text{length}(\gamma_j) = 0$.*

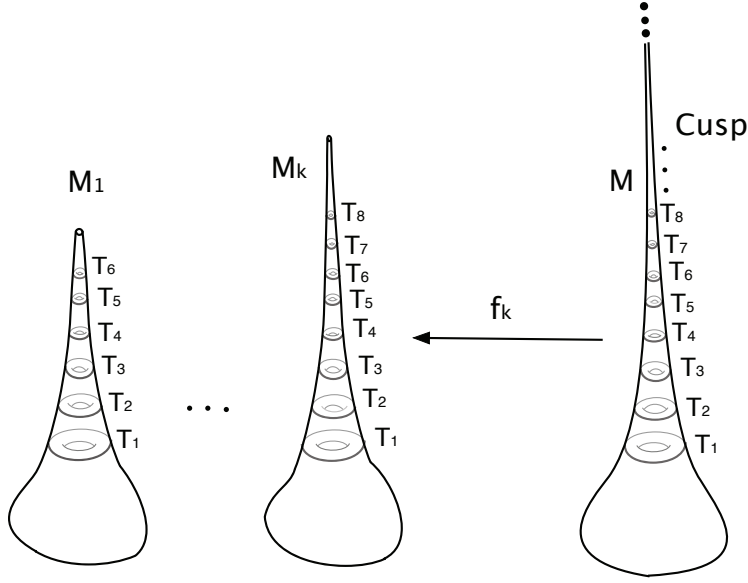


FIGURE 2. Geometric convergence in a cusp

- (4) The horotorus T_1 on the boundary of the cusp of M is carried by f_j to a torus $f_j(T_1)$ that is the boundary of a tubular neighborhood E_j of radius R_j of the geodesic γ_j , with $\lim_{j \rightarrow \infty} R_j = \infty$.

Proof. Thurston showed that a surface bundle over S^1 with fiber a surface of genus $g \geq 2$ is hyperbolic if the monodromy is pseudo-Anosov. A construction of Penner shows that if C and D are two collections of disjoint embedded essential closed curves (with no parallel components) in an oriented surface F that intersect efficiently and fill F , and if ϕ is a product of positive Dehn twists along C and negative Dehn twists along D that twists along each curve at least once, then ϕ is pseudo-Anosov [15]. Take a closed surface F_g of genus $g \geq 2$ and on F a pair of curve collections as above in which $c_1 \in C$ is separating on F . Consider the sequence of pseudo-Anosov maps f_j obtained by composing a fixed number of positive Dehn twists along each curve in C other than c_1 , a fixed number of negative Dehn twists along each curve in D , and k positive twists along c_1 . Then f_j is pseudo-Anosov for each $k \geq 1$, and the manifold M_j formed by constructing a surface bundle over S^1 with monodromy f_j is hyperbolic.

The bundle M_j with monodromy f_j is obtained from M_1 by $1/(k-1)$ -Dehn surgery on c_1 . This surgery first removes a solid-torus neighborhood N of c_1 , giving a hyperbolic 3-manifold with a cusp, M . We call a curve on ∂N lying in a fiber a *longitude* and a curve bounding a disk in N a *meridian*. The surgery attaches a solid-torus $S^1 \times D^2$ to ∂N so that a meridian is mapped to a curve homotopic to one meridian and $(k-1)$ -longitudes. Thurston's Dehn Surgery Theorem shows that the hyperbolic manifolds M_j converge geometrically to M as $j \rightarrow \infty$, and that the core curve of the solid torus attached by the Dehn surgery is isotopic to a closed geodesic $\gamma_j \subset M_j$ and that $\lim_{j \rightarrow \infty} \text{length}(\gamma_j) = 0$. The curve γ_j is null-homologous in M_j , since it is homotopic to a separating curve in a fiber. \square

4. NON-EXISTENCE OF QUASI-AREA-MINIMIZING FIBRATIONS

We now present the proof of Theorem 1.1. We show that with at most finitely many exceptions, the fibers on the hyperbolic manifolds M_j constructed in Lemma 3.1 cannot be isotoped so that all the fibers are (a, b) -quasi-area-minimizing.

We use the notation $M(s)$ to represent the compact manifold formed as the complement in M of $C_{(s, \infty)}$, for each $s \geq 1$, and $M_j(s)$ to represent $f_j(M_s)$. The manifold $M_j(s)$ is a compact manifold with incompressible torus boundary, for j sufficiently large.

Proof. Assume that infinitely many of the manifolds M_j in Lemma 3.1 admit an (a, b) -quasi-area-minimizing fibration. We will obtain a contradiction.

Fix the constant $s_0 = 16\text{Area}(T_1)$. Since the sequence of manifolds $\{M_j\}$ geometrically converges to the cusped hyperbolic manifold M , there is a sequence of radii $r_j \rightarrow \infty$ and diffeomorphisms $f_j : B_{r_j}(M, x) \rightarrow B_{r_j}(M_j, x_j)$ that converge to an isometry on compact subsets of M . In particular, for j sufficiently large the maps $f_j : M(2s_0) \rightarrow M_j(2s_0)$ are diffeomorphisms onto their image that converge to an isometry.

We would like to find a fiber that comes close to $\gamma_j \subset M_j$ but does not intersect γ_j . While such a fiber may not exist, we can find a component of a fiber in a tubular neighborhood of γ_j that has this property.

Claim 4.1. *For j sufficiently large there is a fiber $F_j \subset M_j$ and a component X_j of $F_j \cap f_j(C_{[1, \infty)})$ such that $X_j \cap f_j(T_1) \neq \emptyset$, $X_j \cap f_j(T_{s_0}) \neq \emptyset$, but $X_j \cap f_j(T_{s_0+1}) = \emptyset$. Moreover X_j is either a boundary parallel disk or an essential annulus.*

Proof. The core curve of the Dehn surgery $\gamma_j \subset M_j$ is a null-homologous geodesic in M_j . The torus T_1 maps to the torus $f_j(T_1) \subset M_j$ that bounds a solid torus $E_j \subset M_j$. E_j is a radius R_j tubular neighborhood of γ_j , and the radius of E_j goes to infinity with j .

The surface bundle M_j has an infinite cover \tilde{M}_j whose fundamental group is generated by a fiber. \tilde{M}_j is homeomorphic to the product $F_g \times \mathbb{R}$ of a surface of genus g with \mathbb{R} , and each fiber lifts as a homotopy equivalence. The curve γ_j is homotopic into a fiber, and lifts to a loop $\tilde{\gamma}_j \subset \tilde{M}_j$ homotopic into a fiber of \tilde{M}_j . Therefore there are fibers in \tilde{M}_j that intersect $\tilde{\gamma}_j$ and fibers that are arbitrarily far from $\tilde{\gamma}_j$. By continuity there are fibers in \tilde{M}_j that realize each positive distance from $\tilde{\gamma}_j$. The solid torus E_j also lifts, to a solid torus $\tilde{E}_j \subset \tilde{M}_j$, \tilde{E}_j intersects fibers in \tilde{M}_j in components whose distance from $\tilde{\gamma}_j$ varies from zero to R_j . Let \tilde{X}_j be a component of $\tilde{F}_j \cap \tilde{E}_j$ that intersects a lift of $f_j(T_{s_0}) \subset E_j$ to \tilde{M}_j but not a lift of $f_j(T_{s_0+1})$ and let X_j be the projection of \tilde{X}_j to M_j . Then X_j intersects $f_j(T_1)$ and $f_j(T_{s_0})$ but not $f_j(T_{s_0+1})$, as claimed.

It remains to show that X_j is either a boundary parallel disk or an essential annulus. Suppose first that $\alpha \subset \partial X_j$ is a null homotopic curve on ∂E_j . Since F_j is incompressible, α is the boundary of a disk $D \subset F_j$. We will show that $D \subset E_j$. If not, then D protrudes outside $f_j(C_{[1, \infty)})$ and intersects the interior of $M_j(1)$. Now consider the lift \tilde{D} of D to the cover of M_j given by the subgroup of $\pi_1(M_j)$ generated by γ_j . The disk \tilde{D} has boundary on the boundary of an R_j neighborhood of $\tilde{\gamma}_j$ and has an interior point in the complement of this neighborhood. At a point where it is farthest away from $\tilde{\gamma}_j$ the mean curvature of \tilde{D} is greater than the mean curvature of the boundary of a constant radius tubular neighborhood of the

geodesic $\tilde{\gamma}_j$. The mean curvature of these tubular neighborhood boundaries varies between ∞ and 1, but is always greater than one. Since $D \subset F_j$ has mean curvature $|H| < a < 1$, this gives a contradiction unless $D \subset E_j$.

We now show that X_j is incompressible in E_j . If not, there is a nontrivial compressing disk G for X_j . Since F_j is incompressible, ∂G bounds a disk on F_j , which must protrude out of E_j . But then G intersects ∂E_j in a curve that is not the boundary of a disk in E_j as required by the first case we considered. So X_j is incompressible in E_j and, as with F_j , disjoint from γ_j . Incompressible surfaces in the product of a surface and an interval are either boundary parallel (horizontal) or homotopic to a union of intervals (vertical). It follows that X_j is either a boundary parallel disk or a boundary parallel essential annulus. We have checked that X_j satisfies the claimed properties. \square

We now show that $Y_j = f_j^{-1}(X_j)$ is a $(1,2)$ -quasi-area-minimizing surface in M . The geometric convergence of the M_j to M implies that the areas and the second fundamental forms of the surfaces X_j converge to those of Y_j . Since the mean curvature $|H(X_j)| < a < 1$, it follows that $|H(Y_j)| < 1$, for j sufficiently large.

We now check that Y_j is $(1,2)$ -quasi-area-minimizing for j sufficiently large. Take any compact subsurface $U_j \subset Y_j$ and a surface $V_j \subset M$ homologous to U_j (rel ∂U_j). We need to show $\text{Area}(U_j) \leq 2\text{Area}(V_j)$. It suffices to show this for the area minimizing surface with boundary equal to ∂U_j , so we assume that V_j is such an area minimizing surface. Since V_j is area minimizing and $\partial V_j \subset C_{[1,s_0]}$, Lemma 2.7 implies that $V_j \cap T_{2s_0} = \emptyset$ so that $V_j \subset C_{[1,2s_0]}$. If $\text{Area}(U_j) > 2\text{Area}(V_j)$ then since f_j converge to an isometry on $C_{[1,2s_0]}$, for j large $\text{Area}(f_j(U_j)) > b\text{Area}(f_j(V_j))$, which contradicts the assumption that X_j is (a,b) -quasi-area-minimizing. We conclude that for j sufficiently large Y_j is $(1,2)$ -quasi-area-minimizing.

Now $Y_j \subset C_{[1,2s_0]}$ is either a boundary parallel disk or an essential annulus, and therefore incompressible and separating. So Y_j satisfies the condition of Lemma 2.5, for sufficiently large j , and therefore $Y_j \cap T_s = \emptyset$ for $s \geq s_0$. But the construction of Y_j shows that $Y_j \cap T_{s_0} \neq \emptyset$ for large j , a contradiction. We conclude that at most finitely many of the sequence of manifolds M_j admits a fibration by (a,b) -quasi-area-minimizing fibers, proving Theorem 1.1. \square

5. QUESTIONS

Several questions basic related to this paper remain open.

- (1) Is there a closed hyperbolic 3-manifold with a minimal fibration?
- (2) Is there a closed hyperbolic 3-manifold with a minimal foliation?
- (3) Is there a finite volume hyperbolic 3-manifold that admits a 1-parameter family of complete minimal surfaces? Such a manifold would either be fibered by minimal surfaces or be a union of two I-bundles.
- (4) Is there a closed negatively curved 3-manifold with a minimal fibration?

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, DAVIS CALIFORNIA 95616 &
 SCHOOL OF MATHEMATICS, INSTITUTE FOR ADVANCED STUDY, PRINCETON CA 08540
E-mail address: hass@math.ucdavis.edu